The Local Structure of Nonstandard Representatives of Distributions

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Abstract

It is shown that the nonstandard representatives of Schwartz-distributions, as introduced by K. D. Stroyan and W. A. J. Luxemburg in their book *Introduction to the theory of infinitesimals* [5], are locally equal to a finite-order derivative of a finite-valued and S-continuous function. By 'equality', we mean a pointwise equality, not an equality in a distributional sense. This proves a conjecture by M. Oberguggenberger in [Z. Anal. Anwend. 10 (1991), 263–264]. Moreover, the representatives of the zero-distribution are locally equal to a finite-order derivative of a function assuming only infinitesimal values. These results also unify the nonstandard theory of distributions by K. D. Stroyan and W. A. J. Luxemburg with the theory by R. F. Hoskins and J. Sousa Pinto in [Portugaliae Mathematica 48(2), 195–216].

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1 Stroyan and Luxemburg's theory of distributions

In [5, §10.4], K. D. Stroyan and W. A. J. Luxemburg introduced their nonstandard theory of Schwartz distributions. We give a brief account of the definitions and properties in this theory needed in the sequel. The notations in this section will be used throughout the whole paper (some are different from Stroyan and Luxemburg's). The nonstandard language used is Robinson's.
We will often identify a standard entity A with its image ${}^\sigma A := \{{}^*x : x \in A\}$ when no confusion is possible.
Let Ω be an open subset of $\mathbb{R}^n$. Let $\mathcal{C}^\infty(\Omega)$ be the space of all $\Omega \to \mathbb{C}$-functions possessing continuous derivatives of any order. Let $\mathcal{D}(\Omega)$ be the space of all test-functions on Ω, i.e., all $\mathcal{C}^\infty(\Omega)$-functions with compact support contained in Ω and $\mathcal{D}'(\Omega)$ the space of Schwartz distributions, i.e., continuous linear functionals on $\mathcal{D}(\Omega)$. By $\mathrm{ns}({}^*\Omega)$, we denote the set $\{x \in {}^*\Omega : \exists y \in \Omega : x \approx y\}$ of near-standard points of ${}^*\Omega$. By $\mathrm{Fin}({}^*\mathbb{C})$, we denote the set of finite elements of ${}^*\mathbb{C}$. By st we denote the standard part map.

A topological structure is introduced on ${}^*\mathcal{D}(\Omega)$ in the following way. We denote by ∂^α the partial derivative of order $\alpha \in \mathbb{N}^n$. A function $\phi \in {}^*\mathcal{D}(\Omega)$ is called a *finite* element of ${}^*\mathcal{D}(\Omega)$ iff its support is contained in $\mathrm{ns}({}^*\Omega)$ and if $\partial^\alpha\phi(x) \in \mathrm{Fin}({}^*\mathbb{C})$, for all (finite) multi-indices $\alpha \in \mathbb{N}$ and all $x \in {}^*\Omega$. The set of all finite elements of ${}^*\mathcal{D}(\Omega)$ will be denoted by $\mathrm{Fin}({}^*\mathcal{D}(\Omega))$.
Similarly, $\phi \in {}^*\mathcal{D}(\Omega)$ is called an *infinitesimal* element of ${}^*\mathcal{D}(\Omega)$ iff its support is contained in $\mathrm{ns}({}^*\Omega)$ and if $\partial^\alpha\phi(x) \approx 0$, for all (finite) multi-indices $\alpha \in \mathbb{N}$ and all $x \in {}^*\Omega$. We will write $\phi \approx_{\mathcal{D}} 0$ in this case.
A ${}^*\mathcal{C}^\infty(\Omega)$-function f is called a representative of $T \in \mathcal{D}'(\Omega)$ iff for each $\phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$,

$$\int_{{}^*\Omega} f\phi \approx ({}^*T)(\phi).$$

It can be shown that every function f in the set

$$D'(\Omega) := \left\{ f \in {}^*\mathcal{C}^\infty(\Omega) : \int_{{}^*\Omega} f\phi \in \mathrm{Fin}({}^*\mathbb{C}), \quad \forall \phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega)) \right\}$$

is a representative of a distribution T by means of the definition $T(\phi) := \mathrm{st}\int_{{}^*\Omega} f\phi$. This unique distribution is called the standard part of f and is denoted by $\mathrm{st} f$.
Vice versa, it can be shown that every distribution has a representative in $D'(\Omega)$.
$T \in {}^*\mathcal{D}'(\Omega)$ is called S-continuous iff

$$(\forall \phi \in {}^*\mathcal{D}(\Omega))(\phi \approx_{\mathcal{D}} 0 \implies T(\phi) \approx 0). \tag{1}$$

It can be shown that every $f \in D'(\Omega)$ is S-continuous as an element of ${}^*\mathcal{D}'(\Omega)$. Stroyan and Luxemburg call the elements of $D'(\Omega)$ finite distributions. To avoid the suggestion that $D'(\Omega)$ should be a subset of the space of distributions, and because of the S-continuity as an element of ${}^*\mathcal{D}'(\Omega)$, we will call them *S-distributions* instead.

Remark. A function $f\colon {}^*\Omega \to {}^*\mathbb{C}$ is called S-continuous iff

$$x \approx y \implies f(x) \approx f(y), \qquad \forall x, y \in {}^*\Omega.$$

To avoid confusion for elements of $D'(\Omega)$, we will refer to the S-continuity in the sense of eq. (1) explicitly as 'S-continuity as a linear functional'.

Two elements f, g of $D'(\Omega)$ represent the same distribution iff

$$\int_{{}^*\Omega} f\phi \approx \int_{{}^*\Omega} g\phi, \quad \forall \phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega)).$$

In such case, f and g are called $\mathcal{D}'$-infinitely close, and we write $f \approx_{\mathcal{D}'(\Omega)} g$. If Ω is fixed in the context and no confusion can exist, we often shortly write $f \approx_{\mathcal{D}'} g$.

2 The order of an S-distribution

As it will play a crucial role in proving our results, we recall a theorem about S-continuity which is proved implicitly in [5] (i.e., there is a general theorem on S-continuity from which this theorem follows partly). Also in the context of

Banach spaces, characterizations for S-continuity for internal linear maps are well-known (see e.g. [6]).
We write $K \subset\subset \Omega$ if K is a compact subset of Ω.

Theorem 1. *Let $T \in {}^*\mathcal{D}'(\Omega)$. Then the following are equivalent:*

1. *T is S-continuous*
2. $(\forall \phi \in {}^*\mathcal{D}(\Omega))$ $(\phi \approx_{\mathcal{D}} 0 \implies T(\phi) \in \mathrm{Fin}({}^*\mathbb{C}))$
3. $(\forall \phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega)))$ $(T(\phi) \in \mathrm{Fin}({}^*\mathbb{C}))$
4. $(\forall K \subset\subset \Omega)$ $(\exists C \in \mathbb{R})$ $(\exists m \in \mathbb{N})$ $(\forall \phi \in {}^*\mathcal{D}(K))$
$$\big(\, |T(\phi)| \le C \max_{|\alpha|\le m} \sup_{x \in {}^*K} |\partial^\alpha \phi(x)| \,\big)$$
5. $(\forall K \subset\subset \Omega)$ $(\forall \varepsilon \in \mathbb{R}^+)$ $(\exists \delta \in \mathbb{R}^+)$ $(\exists m \in \mathbb{N})$ $(\forall \phi \in {}^*\mathcal{D}(K))$
$$\big(\max_{|\alpha|\le m} \sup_{x \in {}^*K} |\partial^\alpha \phi(x)| < \delta \implies |T(\phi)| < \varepsilon\big).$$

Proof. $1 \Rightarrow 2 \Rightarrow 3$: follows using the fact that $\varepsilon\phi \approx_{\mathcal{D}} 0$, $\forall \varepsilon \in {}^*\mathbb{R}$ with $\varepsilon \approx 0$ and $\forall \phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$.
$3 \Rightarrow 4$: let $K \subset\subset \Omega$. Let $m \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $\phi \in {}^*\mathcal{D}(K)$. Let
$$M := \max_{|\alpha|\le m} \sup_{x \in {}^*K} |\partial^\alpha \phi(x)| \,.$$
If $M \neq 0$, $\frac{1}{M}\phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$. So $|T(\phi)| = M \underbrace{|T(\phi/M)|}_{\in \mathrm{Fin}({}^*\mathbb{R})}$, and the internal set
$$\big\{m \in {}^*\mathbb{N} : (\forall \phi \in {}^*\mathcal{D}(K))\big(\, |T(\phi)| \le m \max_{|\alpha|\le m} \sup_{x \in {}^*K} |\partial^\alpha \phi(x)| \,\big)\big\}$$
contains all infinite m. By underspill, property 4 holds.
$4 \Rightarrow 5 \Rightarrow 1$: follows using the fact that for each $\phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$, there exists $K \subset\subset \Omega$ such that $\mathrm{supp}\phi \subseteq {}^*K$. $\square$

Following Stroyan and Luxemburg, we introduce the notion of S-distributions of finite order.
An S-distribution f is of *order* at most $m \in \mathbb{N}$ on $K \subset\subset \Omega$ iff
$$(\exists C \in \mathbb{R}^+)(\forall \phi \in {}^*\mathcal{D}(K))\Big(\, \Big|\int_{{}^*\Omega} f\phi\Big| \le C \max_{|\alpha|\le m} \sup_{x \in {}^*K} |\partial^\alpha \phi(x)| \,\Big)$$
or, equivalently, iff
$$(\exists C \in \mathbb{R}^+)(\forall \phi \in \mathrm{Fin}({}^*\mathcal{D}(K)))\Big(\, \Big|\int_{{}^*\Omega} f\phi\Big| \le C \max_{|\alpha|\le m} \sup_{x \in {}^*K} |\partial^\alpha \phi(x)| \,\Big).$$
The smallest $m \in \mathbb{N}$ for which f is of order at most m is (logically) called the order of f.
The equivalence of both definitions follows from the fact that for each $\phi \in {}^*\mathcal{D}(K)$, there exists $M \in {}^*\mathbb{R}^+$ such that $\phi/M \in \mathrm{Fin}({}^*\mathcal{D}(K))$ (see the proof of theorem 1).
Any S-distribution f is of some finite order on any given $K \subset\subset \Omega$. This follows from theorem 1 applied to the 'regular' functional $\phi \mapsto \int_{{}^*\Omega} f\phi \in {}^*\mathcal{D}'(\Omega)$.

3 Introduction to the new results in this paper

In their short section on distributions (which they call a 'sketch' themselves), Stroyan and Luxemburg only mention S-distributions of finite order for proving the theorem that every distribution is locally a finite order derivative of a continuous function, by means of the fact (mentioned as an exercise) that any S-distribution of finite order is $\mathcal{D}'$-infinitely close to a finite-order derivative of an S-continuous function $\in D'(\Omega)$. We will show that the order of an S-distribution f is *not* equal to the order of the distribution $\mathrm{st} f$. The difference between these two orders will be the key to give (at least partially) an answer the following questions.

What do S-distributions look like? Except from their definition, what are qualitative ways in which they differ from ordinary functions in ${}^*\mathcal{C}^\infty(\Omega)$?

How much can two representatives of the same distribution differ? Except from the fact that they are $\mathcal{D}'$-infinitely close, are there qualitative ways in which this difference can be described?

It may be clear from the following example that there is hardly any pointwise way in which different representations from a given distribution coincide in general.

Example. For each $k \in \mathbb{Z}$ and $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$, the function $\omega^k \sin(\omega x) \in {}^*\mathcal{C}^\infty(\mathbb{R})$ is a representative of the zero-distribution ($\in \mathcal{D}'(\mathbb{R})$).

Proof. For $k < 0$, $f_k(x) = \omega^k \sin(\omega x) \approx 0$, $\forall x \in {}^*\mathbb{R}$, so $f_k \approx_{\mathcal{D}'} 0$. As it is well-known that the distributional derivatives coincide with the derivatives of the representatives, also the second derivative $f_k'' = -f_{k+2} \approx_{\mathcal{D}'} 0$. Inductively, $f_k \approx_{\mathcal{D}'} 0$, $\forall k \in \mathbb{N}$. $\square$

In the example, the method to find heavily irregular representatives of the zero-distribution was by taking derivatives of a function that assumes infinitesimal values. We will prove that no other irregularities can exist, i.e., that every $f \approx_{\mathcal{D}'} 0$ is (locally) pointwise equal to some finite order derivative of a ${}^*\mathcal{C}^\infty(\Omega)$-function assuming only infinitesimal values.

Similarly, we will prove that every $f \in D'(\Omega)$ is (locally) pointwise equal to some finite order derivative of an S-continuous and finite-valued ${}^*\mathcal{C}^\infty(\Omega)$-function.

The last of these two assertions was already mentioned (for $\Omega = \mathbb{R}^n$ and omitting the S-continuity) in [3, Prop. 2.10] in the nonstandard language of Nelson, but, as it appears from the correction to [3], it still remained unproved.

Although such theorems are of a fashion similar to the classical local representation theorem of distributions, the distributional order cannot be a measure for the order of the derivative in our representation theorems: already for the zero-distribution, which is trivially of order 0, the order of the derivative may be arbitrary large.

4 Proofs of the new results

First, we point out more explicitly that the order of an S-distribution is not equal to the distributional order of its standard part. For x, $y \in {}^*\mathbb{R}$, we write $x \lessapprox y$ iff $x < y$ or $x \approx y$.

Theorem 2. *Let $f \in D'(\Omega)$ and $K \subset\subset \Omega$. Then the (distributional) order of* $\mathrm{st} f$ *on K is the smallest $m \in \mathbb{N}$ such that*

$$(\exists C \in \mathbb{R}^+)(\forall \phi \in \mathrm{Fin}({}^*\mathcal{D}(K)))\Big(\left| \int_{^*\Omega} f\phi \right| \lessapprox C \max_{|\alpha|\le m} \sup_{x\in {}^*K} |\partial^\alpha \phi(x)| \Big). \qquad (2)$$

Proof. 1. Let the order of $T := \mathrm{st} f$ on K be at most m, i.e. (by transfer),

$$(\exists C \in \mathbb{R}^+)(\forall \phi \in {}^*\mathcal{D}(K))(|{}^*T(\phi)| \le C \max_{|\alpha|\le m} \sup_{x\in {}^*K} |\partial^\alpha \phi(x)|).$$

Since for $\phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$, ${}^*T(\phi) \approx \int_{^*\Omega} f\phi$, we find that formula (2) holds for this m.
2. On the other hand, suppose that formula (2) holds for some $m \in \mathbb{N}$. Again by the fact that for $\phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$, ${}^*T(\phi) \approx \int_{^*\Omega} f\phi$ (with $T = \mathrm{st} f$), we have in particular that

$$(\exists C \in \mathbb{R}^+)(\forall \phi \in \mathcal{D}(K))(|{}^*T({}^*\phi)| \lessapprox C \max_{|\alpha|\le m} \sup_{x\in {}^*K} |\partial^{\alpha} {}^*\phi(x)|).$$

Since both sides of the $\lessapprox$-inequality are standard numbers, we actually have a $\le$-inequality, and the (distributional) order of T on K is at most m. □

Corollary. *The order of an S-distribution f is at least the distributional order of* $\mathrm{st} f$.

The following example shows that the difference of the two orders can be arbitrary large.

Example. Consider $f(x) = \omega^k \sin(\omega x)$, with $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$. It has order k on every compact $K \subset\subset \mathbb{R}$. On the other hand, $f \approx_{\mathcal{D}'} 0$ (see example 3), so the order of the corresponding standard distribution is 0.

Proof. Let $\phi \in {}^*\mathcal{D}(K)$. For some $R \in \mathbb{R}$, $K \subseteq [-R, R]$. Then by partial integration,

$$\int_{^*\mathbb{R}} f\phi = (-1)^k \int_{^*\mathbb{R}} g(x)\phi^{(k)}(x)\, dx$$

with $g^{(k)} = f$, so we can choose $g(x) \in \{\pm \sin(\omega x), \pm \cos(\omega x)\}$. So

$$\left| \int_{^*\mathbb{R}} f\phi \right| \le 2R \sup_{x\in {}^*K} |g(x)| \sup_{x\in {}^*K} \left|\phi^{(k)}(x)\right| \le 2R \sup_{x\in {}^*K} \left|\phi^{(k)}(x)\right|,$$

so the order is at most k.
To see that the order is at least k, let $\phi_0 \in \mathcal{D}(K)$ with $\int \phi_0 = 1$ and let $\phi(x) := \sin(\omega x)\phi_0(x)$. Then

$$\frac{1}{\omega^k} \int_{^*\mathbb{R}} f\phi = \frac{1}{2} \int_{^*\mathbb{R}} (1 - \cos(2\omega x))\phi_0(x)\, dx \approx \frac{1}{2},$$

since $\cos(2\omega x) \approx_{\mathcal{D}'} 0$ (similarly as in example 3). On the other hand, for each $j \in \mathbb{N}$, $\sup_{x\in {}^*K} \left|\phi^{(j)}(x)\right| \le M\omega^j$ for some $M \in \mathbb{R}$, so for this $\phi \in {}^*\mathcal{D}(K)$, $\left|\int_{^*\mathbb{R}} f\phi\right| > C \max_{j\le k-1} \sup_{x\in {}^*K} \left|\phi^{(j)}(x)\right|$, $\forall C \in \mathbb{R}$. □

Next, we will prepare our main results. First, we show that distributional anti-derivatives can be dealt with on representatives. To our knowledge, such a theorem is not available in the nonstandard literature. Just for convenience, we only deal with partial derivatives in the first variable.
We introduce the following notation: for $x = (x_1, \ldots, x_n) \in {}^*\mathbb{R}^n$, we will write $\tilde{x}_i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Similarly, for $i < j$, $\tilde{x}_{i,j} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ and so on for $\tilde{x}_{i,j,k}, \ldots$

Lemma 3. *Let Ω be an open interval (i.e., it is the Cartesian product of n one-dim. intervals). Let $T \in \mathcal{D}'(\Omega)$ and f a representative of T. Then there exists an S-distribution $g \in D'(\Omega)$ with $\partial_1 g = f$. As a consequence, g determines a distribution U with $\partial_1 U = T$.*

Proof. 1. In order to get some insight in the proof, we first consider the one-dimensional case.
Choose $F \in {}^*\mathcal{C}^\infty(\Omega)$ such that $F' = f$ on Ω. We can only expect F to be an S-distribution if the integration constant is well-chosen. So, we seek $C \in {}^*\mathbb{C}$ such that $\int_{^*\mathbb{R}} (F + C)\phi \in \mathrm{Fin}({}^*\mathbb{C})$, $\forall \phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$. Now fix $\phi_0 \in \mathcal{D}(\Omega)$, with $\int_{\mathbb{R}} \phi_0 = 1$. Then the previous condition specifies to $\int_{^*\mathbb{R}} F {}^*\phi_0 + C \in \mathrm{Fin}({}^*\mathbb{C})$. As a finite change in the constant doesn't influence the S-distributional character of $F + C$, we can put $C := -\int_{^*\mathbb{R}} F {}^*\phi_0$. Then, for any $\phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$,

$$\int_{^*\mathbb{R}} (F + C)\phi = \int_{^*\mathbb{R}} F(t) \underbrace{\Big(\phi(t) - \Big(\int_{^*\mathbb{R}} \phi \Big) {}^*\phi_0(t) \Big)}_{=:\psi(t) \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))} \, dt.$$

As $\int_{^*\mathbb{R}} \psi = 0$, $\psi^{(-1)}(x) := \int_{-\infty}^{x} \psi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$, and by partial integration,

$$\int_{^*\mathbb{R}} (F + C)\phi = -\int_{^*\mathbb{R}} f \psi^{(-1)} \in \mathrm{Fin}({}^*\mathbb{C}),$$

since f is an S-distribution.
2. In the general case, we choose an arbitrary anti-derivative F of f in the first variable (on Ω). E.g., if $\Omega = (a_1, b_1) \times \cdots \times (a_n, b_n)$ (a_i, $b_i \in \mathbb{R} \cup \{-\infty, +\infty\}$), then for any $a_1 < c < b_1$, $\int_c^{x_1} f(t, \tilde{x}_1)\, dt$ is such an anti-derivative). An anti-derivative is determined up to a function $G(\tilde{x}_1)$. Now it turns out that, for a fixed $\phi_0 \in \mathcal{D}((a_1, b_1))$ with $\int_{\mathbb{R}} \phi_0 = 1$, $G(\tilde{x}_1) = -\int_{^*\mathbb{R}} F(t, \tilde{x}_1) {}^*\phi_0(t)\, dt$ is a good choice: for any $\phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$,

$$\int_{^*\mathbb{R}^n} (F(x) + G(\tilde{x}_1))\phi(x)\, dx$$
$$= \int_{^*\mathbb{R}^n} F(x) \underbrace{\Big(\phi(x) - \Big(\int_{^*\mathbb{R}} \phi(u, \tilde{x}_1)\, du \Big) {}^*\phi_0(x_1) \Big)}_{=:\psi(x)} \, dx.$$

As Ω is an interval, $\psi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$. Moreover, $\int_{^*\mathbb{R}} \psi(t, \tilde{x}_1)\, dt = 0$, $\forall \tilde{x}_1 \in {}^*\mathbb{R}^{n-1}$, so $\chi(x) := \int_{-\infty}^{x_1} \psi(t, \tilde{x})\, dt \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$ and similarly as in the one-dimensional case, we find that $\int_{^*\mathbb{R}^n} (F(x) + G(\tilde{x}_1))\phi(x)\, dx \in \mathrm{Fin}({}^*\mathbb{C})$. $\square$

Lemma 4. *Let $f \in D'(\Omega)$ of order $\le m$ on an interval $K \subset\subset \Omega$, $m > 0$. Then there exists $g \in D'(\Omega)$ of order $\le m - 1$ on K such that $\partial_1 \cdots \partial_n g = f$ on ${}^*K$.*

Proof. Let $K = [a_1, b_1] \times \cdots \times [a_n, b_n]$. We will show that, if f satisfies

$$\left|\int_{{}^*\Omega} f\phi\right| \le C \sup_{x\in{}^*K} \left|\partial^{(k,\alpha)}\phi(x)\right|, \quad \forall \phi \in \mathrm{Fin}({}^*\mathcal{D}(K))$$

for some $C \in \mathbb{R}$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{n-1}$, then the anti-derivative $g(x) = F(x) + G(\tilde{x}_1)$ in the first variable defined in lemma 3 satisfies

$$\left|\int_{{}^*\Omega} g\phi\right| \le C' \max_{j\le l} \sup_{x\in{}^*K} \left|\partial^{(j,\alpha)}\phi(x)\right|, \quad \forall \phi \in \mathrm{Fin}({}^*\mathcal{D}(K))$$

with $C' \in \mathbb{R}$ and $l = \max(k-1, 0)$.
Let $\phi \in \mathrm{Fin}({}^*\mathcal{D}(K))$. With ψ, $\chi \in \mathrm{Fin}({}^*\mathcal{D}(K))$ as in lemma 3, we have

$$\left|\int_{{}^*\Omega} g\phi\right| = \left|\int_{{}^*\Omega} f\chi\right| \le C \sup_{x\in{}^*K} \left|\partial^{(k,\alpha)}\chi(x)\right| = C \sup_{x\in{}^*K} \left|\partial^{(0,\alpha)}\partial_1^k\partial_1^{-1}\psi(x)\right|.$$

In case $k = 0$, we have for $x \in {}^*K$ that

$$\left|\partial^{(0,\alpha)}\partial_1^{-1}\psi(x)\right| = \left|\int_{-\infty}^{x_1} \partial^{(0,\alpha)}\psi(t_1, \tilde{x}_1)\,dt_1\right| \le (b_1 - a_1) \sup_{x\in{}^*K} \left|\partial^{(0,\alpha)}\psi(x)\right|,$$

so in any case we have (for some C', $C'' \in \mathbb{R}$, independent of ϕ)

$$\begin{aligned}\left|\int_{{}^*\Omega} g\phi\right| &\le C' \sup_{x\in{}^*K} \left|\partial^{(l,\alpha)}\phi(x)\right| + C' \sup_{x\in{}^*K} \left|D^l\, {}^*\phi_0(x_1) \int_{{}^*\mathbb{R}} \partial^{(0,\alpha)}\phi(u, \tilde{x}_1)\,du\right| \\ &\le C'' \max_{j\le l} \sup_{x\in{}^*K} \left|\partial^{(j,\alpha)}\phi(x)\right|.\end{aligned}$$

Since g is well-defined on ${}^*\Omega'$, for some interval $\Omega' \subseteq \Omega$ with $K \subset\subset \Omega'$, we can use $\phi_0 \in \mathcal{D}(\Omega')$ with $\phi_0 = 1$ on K to ensure that $g^*\phi_0 \in D'(\Omega)$ without changing the values on ${}^*K$.
If we repeatedly apply also the analogous result for the variables $x_2, \ldots, x_n$, we finally conclude that the order of the primitive $(\partial_1 \cdots \partial_n)^{-1} f$ has decreased (if $m > 0$). □

For $K \subset\subset \Omega$, we call $L^\infty(K)$ the space of all (standard) bounded and (Lebesgue-)measurable functions $f\colon \Omega \to \mathbb{C}$ with support contained in K.

Lemma 5. *Let $K \subset\subset \Omega$ an interval. An S-distribution f is of order zero on K iff*

$$(\exists C \in \mathbb{R}^+)(\forall \phi \in {}^*L^\infty(K))\Big(\left|\int_{{}^*\Omega} f\phi\right| \le C \sup_{x\in{}^*K} |\phi(x)|\Big).$$

Proof. Let $f \in \mathcal{C}^\infty(\Omega)$ and $\phi \in L^\infty(K)$. Then by a classical density theorem, it is clear that there exists some $h \in \mathcal{D}(K)$ such that

$$\left|\int_{\Omega} f\phi - \int_{\Omega} fh\right| \le \sup_{x\in K} |\phi(x)| \quad \& \quad \sup_{x\in K} |h(x)| \le 2 \sup_{x\in K} |\phi(x)|.$$

By transfer, we have $(\forall f \in {}^*\mathcal{C}^\infty(\Omega))$ $(\forall \phi \in {}^*L^\infty(K))$ $(\exists h \in {}^*\mathcal{D}(K))$

$$\Big(\left|\int_{{}^*\Omega} f\phi - \int_{{}^*\Omega} fh\right| \le \sup_{x\in{}^*K} |\phi(x)| \quad \& \quad \sup_{x\in{}^*K} |h(x)| \le 2 \sup_{x\in{}^*K} |\phi(x)|\Big).$$

If in particular f is an S-distribution of order 0 on K, then

$$(\exists C \in \mathbb{R}^+)(\forall h \in {}^*\mathcal{D}(K))\Big(\Big|\int_{^*\Omega} fh\Big| \le C \sup_{x\in {}^*K} |h(x)|\Big).$$

The result follows by combining these two formulas. $\square$

Lemma 6. *Let $f \in D'(\Omega)$. Suppose that f is of order zero on a (standard) interval $K = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset\subset \Omega$. Then*

1. *there exists $g \in {}^*\mathcal{C}^\infty(\Omega)$ which is bounded on ${}^*K$ by a standard constant and such that $\partial_1 \cdots \partial_n g = f$ on ${}^*K$.*
2. *there exists $h \in {}^*\mathcal{C}^\infty(\Omega)$ which is S-continuous and bounded by a standard constant on ${}^*K$ and such that $\partial_1^2 \cdots \partial_n^2 h = f$ on ${}^*K$.*

Proof. 1. Let $x = (x_1, \dots, x_n)$ and $t = (t_1, \dots, t_n)$. For $A \subset \Omega$, we denote the characteristic function of A by χ_A. Then (for $x \in {}^*K$)

$$g(x) := \int_{a_1}^{x_1} dt_1 \cdots \int_{a_n}^{x_n} f(t)\, dt_n = \int_{^*\Omega} f\chi_{[a_1,x_1]\times\cdots\times[a_n,x_n]}$$

clearly satisfies $\partial_1 \cdots \partial_n g = f$ on ${}^*K$. Further, applying the previous lemma with $\phi = \chi_{[a_1,x_1]\times\cdots\times[a_n,x_n]} \in {}^*L^\infty(K)$ (if $x \in {}^*K$), we find $C \in \mathbb{R}^+$ such that

$$(\forall x \in {}^*K)\big(|g(x)| \le C \underbrace{\sup_{x\in {}^*K} |\phi(x)|}_{=1}\big).$$

2. If g satisfies the conditions from part 1, then (for $x \in {}^*K$)

$$h(x) := \int_{a_1}^{x_1} dt_1 \cdots \int_{a_n}^{x_n} g(t)\, dt_n$$

clearly satisfies $\partial_1^2 \cdots \partial_n^2 h = f$ on ${}^*K$. Further, for $\varepsilon \approx 0$, $\varepsilon > 0$,

$$|h(x_1 + \varepsilon, \tilde{x}_1) - h(x)| = \left|\int_{x_1}^{x_1+\varepsilon} dt_1 \cdots \int_{a_n}^{x_n} g(t)\, dt_n\right| \le C\varepsilon \prod_{i\ne 1}(b_i - a_i) \approx 0$$

and similarly for the other variables. So $h(x) \approx h(y)$ as soon as $x \approx y$ (x, $y \in {}^*K$). Further, $|h(x)| \le C \prod_i (b_i - a_i) \in \mathrm{Fin}({}^*\mathbb{C})$, $\forall x \in {}^*K$. $\square$

We are now ready to prove the first main result.

Theorem 7. *Let $f \in {}^*\mathcal{C}^\infty(\Omega)$. Then $f \in D'(\Omega)$ iff for each $K \subset\subset \Omega$, there exists a $g \in D'(\Omega)$ which is finite-valued and S-continuous on ${}^*K$ and such that f is a finite order derivative of g on ${}^*K$.*

Proof. $\Leftarrow$: follows using the fact that for each $\phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$, there exists $K \subset\subset \Omega$ such that $\mathrm{supp}\phi \subseteq {}^*K$.
$\Rightarrow$: 1. We first consider the special case where $K \subset\subset \Omega$ is an interval.
Take an interval $K' \subset\subset \Omega$ with $K \subset\subset {}^\circ(K')$, the (topological) interior of K'. Since f has a finite order m on K', we find, by repeatedly applying lemma 4, some $\tilde{g} \in D'(\Omega)$ of order zero on K' such that $(\partial_1 \cdots \partial_n)^m \tilde{g} = f$ on ${}^*K'$. By

lemma 6, we find $h \in {}^*\mathcal{C}^\infty(\Omega)$ which is finite and S-continuous on ${}^*K'$ and such that $(\partial_1 \cdots \partial_n)^{m+2} h = f$ on ${}^*K'$. If $\phi_0 \in \mathcal{D}(K')$ with $\phi_0 = 1$ on K, then $g := h {}^*\phi_0 \in D'(\Omega)$ has the required properties.
2. We consider the special case where $f(x) = 0$, $\forall x \notin \mathrm{ns}({}^*\Omega)$.
Then f can be extended to a ${}^*\mathcal{C}^\infty(\mathbb{R}^n)$-function, setting $f(x) := 0$ if $x \in {}^*\mathbb{R}^n \setminus \mathrm{ns}({}^*\Omega)$. We claim that this extension $\in D'(\mathbb{R}^n)$. There exists $K_0 \subset\subset \Omega$ such that $f(x) = 0$ outside ${}^*K_0$. Choose $\phi_0 \in \mathcal{D}(\Omega)$ with $\phi_0 = 1$ on K_0. Then for any $\phi \in \mathrm{Fin}({}^*\mathcal{D}(\mathbb{R}^n))$,

$$\int_{{}^*\mathbb{R}^n} f\phi = \int_{{}^*\Omega} f \underbrace{{}^*\phi_0 \phi}_{\in \mathrm{Fin}({}^*\mathcal{D}(\Omega))} \in \mathrm{Fin}({}^*\mathbb{C}).$$

Now let $K \subset\subset \Omega$ arbitrarily. Since $K \subseteq L \subset\subset \mathbb{R}^n$, with L an interval (possibly $L \not\subseteq \Omega$), we conclude from part 1 that there exists a $g \in D'(\mathbb{R}^n)$ which is finite and S-continuous on ${}^*L$ and such that (the extended) f is a finite order derivative of g on ${}^*L$. The restriction of g to ${}^*\Omega$ has the required properties.
3. In the general case, let $K \subset\subset \Omega$. Taking $\phi_0 \in \mathcal{D}(\Omega)$ with $\phi_0 = 1$ on K, we apply part 2 on $f {}^*\phi_0 \in D'(\Omega)$. □

The second main result will follow from the previous theorem together with some additional lemmas.

Lemma 8. *Let $\Omega = (a_1, b_1) \times \cdots (a_n, b_n) \subseteq \mathbb{R}^n$ be an open interval (possibly $a_i = -\infty$, $b_i = +\infty$). Let $\tilde{\Omega} := (a_2, b_2) \times \cdots (a_n, b_n) \subseteq \mathbb{R}^{n-1}$. Let $f \in {}^*\mathcal{C}^\infty(\Omega)$ be independent of x_1, so it can be identified with a ${}^*\mathcal{C}^\infty(\tilde{\Omega})$-function. Then*

1. $f(\tilde{x}_1) \in D'(\Omega) \iff f(\tilde{x}_1) \in D'(\tilde{\Omega})$.
2. $f(\tilde{x}_1) \approx_{\mathcal{D}'(\Omega)} 0 \iff f(\tilde{x}_1) \approx_{\mathcal{D}'(\tilde{\Omega})} 0$.

As a consequence, the expression $f(\tilde{x}_1) \approx_{\mathcal{D}'} 0$ is unambiguous.

Proof. 1. $\Rightarrow$: Let $f(\tilde{x}_1) \in D'(\Omega)$. Fix $\psi(x_1) \in \mathrm{Fin}({}^*\mathcal{D}(a_1, b_1))$ with $\int_{{}^*\mathbb{R}} \psi = 1$. Choose $\phi(\tilde{x}_1) \in \mathrm{Fin}({}^*\mathcal{D}(\tilde{\Omega}))$ arbitrarily. Then $\psi(x_1)\phi(\tilde{x}_1) \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$, so

$$\mathrm{Fin}({}^*\mathbb{C}) \ni \int_{{}^*\Omega} f(\tilde{x}_1)\psi(x_1)\phi(\tilde{x}_1)\, dx = \underbrace{\int_{a_1}^{b_1} \psi(x_1)\, dx_1}_{=1} \int_{{}^*\tilde{\Omega}} f(\tilde{x}_1)\phi(\tilde{x}_1)\, d\tilde{x}_1,$$

which means that $f(\tilde{x}_1) \in D'(\tilde{\Omega})$.
$\Leftarrow$: Let $f(\tilde{x}_1) \in D'(\tilde{\Omega})$. For any $\phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$ and $c \in \mathrm{ns}({}^*(a_1, b_1))$, the map $\tilde{x}_1 \mapsto \phi(c, \tilde{x}_1) \in \mathrm{Fin}({}^*\mathcal{D}(\tilde{\Omega}))$, so

$$\psi(c) := \int_{{}^*\tilde{\Omega}} f(\tilde{x}_1)\phi(c, \tilde{x}_1)\, d\tilde{x}_1 \in \mathrm{Fin}({}^*\mathbb{C}).$$

Further, for some $K \subset\subset (a_1, b_1)$, if c lies outside ${}^*K$, $\psi(c) = 0$. So

$$\int_{{}^*\Omega} f(\tilde{x}_1)\phi(x)\, dx = \int_{{}^*K} \psi(x_1)\, dx_1 \in \mathrm{Fin}({}^*\mathbb{C}),$$

which means that $f(\tilde{x}_1) \in D'(\Omega)$.
2. Similar. □

The following lemmas could be considered as exercises in distribution theory. To our knowledge, they are not widely known. Therefore, we will include a nonstandard version with proof.

Lemma 9. *Let Ω be an open interval.*
If $f \in D'(\Omega)$ and $\partial^\alpha f \approx_{\mathcal{D}'} 0$, then there exist $g_{ij} \in D'(\Omega)$ such that

$$f(x) \approx_{\mathcal{D}'} \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} g_{ij}(\tilde{x}_i) x_i^j.$$

Proof. 1. We first show that, if $F \in D'(\Omega)$ and $\partial_1 F \approx_{\mathcal{D}'} 0$, then F is $\mathcal{D}'$-infinitely close to a $D'(\Omega)$-function which doesn't depend on x_1.
If we choose $G(\tilde{x}_1)$ as in lemma 3, we see that for all $\phi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$,

$$\int_{{}^*\mathbb{R}^n} (F(x) + G(\tilde{x}_1))\phi(x)\, dx = \int_{{}^*\mathbb{R}^n} (\partial_1 F)(x)\chi(x)\, dx \approx 0$$

with $\chi \in \mathrm{Fin}({}^*\mathcal{D}(\Omega))$ as in lemma 3. So $F(x) \approx_{\mathcal{D}'} -G(\tilde{x}_1)$.
2. Now suppose that $f \in D'(\Omega)$ and

$$\partial_1 f(x) \approx_{\mathcal{D}'} \sum_{i=1}^{n} \sum_{j=0}^{m_i} g_{ij}(\tilde{x}_i) x_i^j \tag{3}$$

for some $g_{ij} \in D'(\Omega)$. We will show that

$$f(x) \approx_{\mathcal{D}'} \sum_{i=1}^{n} \sum_{j=0}^{\tilde{m}_i} \tilde{g}_{ij}(\tilde{x}_i) x_i^j$$

for some $\tilde{g}_{ij} \in D'(\Omega)$, $\tilde{m}_1 = m_1 + 1$, $\tilde{m}_2 = m_2$, $\ldots$, $\tilde{m}_n = m_n$.
We notice that the right-hand side of eq. (3) is equal to

$$\partial_1 \left(\sum_{j=0}^{m_1} g_{1j}(\tilde{x}_1) \frac{x_1^{j+1}}{j+1} + \sum_{i=2}^{n} \sum_{j=0}^{m_i} (\partial_1^{-1} g_{ij})(\tilde{x}_i) x_i^j \right).$$

From the explicit construction of the primitives $\partial_1^{-1} g_{ij}$ in lemma 3, it is immediate that also they are independent of x_i. Then applying part 1 on the difference of both sides in eq. (3), we find that there exists $G(\tilde{x}_1) \in D'(\Omega)$ such that

$$f(x) \approx_{\mathcal{D}'} G(\tilde{x}_1) + \sum_{j=0}^{m_1} g_{1j}(\tilde{x}_1) \frac{x_1^{j+1}}{j+1} + \sum_{i=2}^{n} \sum_{j=0}^{m_i} (\partial_1^{-1} g_{ij})(\tilde{x}_i) x_i^j$$

which has the required form.
3. Now the theorem follows inductively using part 2 and the analogous formulas for the other variables ($\neq 1$), also using the fact that if $f \in D'(\Omega)$, then $\partial^\beta f \in D'(\Omega)$, $\forall \beta \in \mathbb{N}^n$. ∎

Lemma 10. *Let Ω be an open interval. Let $f \in {}^*\mathcal{C}^\infty(\Omega)$ be S-continuous and finite-valued on $\mathrm{ns}({}^*\Omega)$ and suppose that $\partial^\alpha f \approx_{\mathcal{D}'} 0$. Then there exist $g_{ij} \in {}^*\mathcal{C}^\infty(\Omega)$ which are S-continuous and finite-valued on $\mathrm{ns}({}^*\Omega)$ such that*

$$f(x) \approx_{\mathcal{D}'} \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} g_{ij}(\tilde{x}_i) x_i^j.$$

Proof. First notice that a ${}^*\mathcal{C}^\infty(\Omega)$-function which is finite-valued on $\mathrm{ns}({}^*\Omega)$, is $\in D'(\Omega)$. Let $\Omega = (a_1, b_1) \times \cdots \times (a_n, b_n)$ and $\tilde{\Omega} := (a_2, b_2) \times \cdots \times (a_n, b_n)$. Let $\partial^\alpha f \approx_{\mathcal{D}'} 0$ and let $\alpha =: (\alpha_1, \tilde{\alpha})$, $\tilde{\alpha} \in \mathbb{N}^{n-1}$. By the previous lemma,

$$f(x) \approx_{\mathcal{D}'} \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} h_{ij}(\tilde{x}_i) x_i^j, \tag{4}$$

with $h_{ij} \in D'(\Omega)$. Now consider $c \in \mathrm{ns}^*(a_1, b_1)$ arbitrarily. Fix $\psi(x_1) \in \mathcal{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \psi = 1$ and $\psi \geq 0$. Let $\psi_m(x_1) := m\psi(mx_1)$, $\forall m \in {}^*\mathbb{N}$. Let $\phi(\tilde{x}_1) \in \mathrm{Fin}({}^*\mathcal{D}(\tilde{\Omega}))$ arbitrarily. Since

$$\partial^{(0,\tilde{\alpha})} f(x) \approx_{\mathcal{D}'} \sum_{j=0}^{\alpha_1 - 1} \partial^{\tilde{\alpha}} h_{1j}(\tilde{x}_1) x_1^j,$$

we have for sufficiently large $m \in \mathbb{N}$ (such that $\mathrm{supp}(\psi_m(c - x_1)) \subset \mathrm{ns}^*(a_1, b_1)$) that

$$\int_{{}^*\Omega} \partial^{(0,\tilde{\alpha})} f(x) \psi_m(c - x_1) \phi(\tilde{x}_1)\, dx \approx \sum_{j=0}^{\alpha_1 - 1} \int_{{}^*\Omega} \partial^{\tilde{\alpha}} h_{1j}(\tilde{x}_1) x_1^j \psi_m(c - x_1) \phi(\tilde{x}_1)\, dx. \tag{5}$$

By Robinson's sequential lemma, this also holds for some $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$. If $\tilde{x}_1 \in \mathrm{ns}({}^*\tilde{\Omega})$, the map $x_1 \to f(x)$ is S-continuous on $\mathrm{ns}({}^*(a_1, b_1))$. Then $\forall \tilde{x}_1 \in \mathrm{ns}({}^*\tilde{\Omega})$,

$$\begin{aligned} &\left| \int_{a_1}^{b_1} f(x) \psi_\omega(c - x_1)\, dx_1 - f(c, \tilde{x}_1) \right| \\ &= \left| \int_{a_1}^{b_1} (f(x) - f(c, \tilde{x}_1)) \psi_\omega(c - x_1)\, dx_1 \right| \leq \sup_{x_1 \in \mathrm{supp}\psi_\omega} |f(x) - f(c, \tilde{x}_1)| \approx 0 \end{aligned}$$

since $\mathrm{supp}\psi_\omega$ contains only infinitesimals and $\int_{{}^*\mathbb{R}} |\psi_\omega| = 1$. In particular, they are $\mathcal{D}'$-infinitely close. So also

$$\int_{a_1}^{b_1} \partial^{(0,\tilde{\alpha})} f(x) \psi_\omega(c - x_1)\, dx_1 \approx_{\mathcal{D}'} \partial^{\tilde{\alpha}} f(c, \tilde{x}_1).$$

On the other hand,

$$\begin{aligned} &\int_{{}^*\Omega} \partial^{\tilde{\alpha}} h_{1j}(\tilde{x}_1) x_1^j \psi_\omega(c - x_1) \phi(\tilde{x}_1)\, dx \\ &\qquad = \underbrace{\int_{{}^*\tilde{\Omega}} \partial^{\tilde{\alpha}} h_{1j}(\tilde{x}_1) \phi(\tilde{x}_1)\, d\tilde{x}_1}_{\in \mathrm{Fin}({}^*\mathbb{C})} \underbrace{\int_{a_1}^{b_1} x_1^j \psi_\omega(c - x_1)\, dx_1}_{\approx c^j}, \end{aligned}$$

so we find from equation (5) that for each $c \in \mathrm{ns}({}^*(a_1, b_1))$

$$\partial^{\tilde{\alpha}} f(c, \tilde{x}_1) \approx_{\mathcal{D}'} \sum_{j=0}^{\alpha_1 - 1} c^j \partial^{\tilde{\alpha}} h_{1j}(\tilde{x}_1).$$

Now choose α_1 different values $c_i \in \mathrm{ns}({}^*(a_1, b_1))$, with $c_i \not\approx c_j$ if $i \neq j$. Then we find a linear system with α_1 equations and α_1 unknown functions $\partial^{\tilde{\alpha}} h_{1j}$. The

determinant of the system is a Vandermonde-determinant equal to $\prod_{i<j}(c_j - c_i) \not\approx 0$. Therefore, each $\partial^{\tilde{\alpha}} h_{1j}(\tilde{x}_1)$ is $\mathcal{D}'$-infinitely close to a Fin(${}^*\mathbb{C}$)-linear combination of the $\partial^{\tilde{\alpha}} f(c_j, \tilde{x}_1)$, which we call $\partial^{\tilde{\alpha}} g_{1j}(\tilde{x}_1)$. So $g_{1j}(\tilde{x}_1) \in {}^*\mathcal{C}^\infty(\Omega)$ are S-continuous and finite-valued on ns(${}^*\Omega$). By the previous lemma (applied to $\tilde{\Omega} \subseteq \mathbb{R}^{n-1}$),

$$h_{1j}(\tilde{x}_1) \approx_{\mathcal{D}'} g_{1j}(\tilde{x}_1) + \sum_{i=2}^{n} \sum_{k=0}^{\alpha_i - 1} \tilde{h}_{ik}(\tilde{x}_{1i}) x_i^k,$$

for some $\tilde{h}_{ik} \in D'(\Omega)$. Substituting these expressions, together with the analogous expressions for $h_{ij}(\tilde{x}_i)$ (with $i > 1$), in formula (4) yields that

$$f(x) \approx_{\mathcal{D}'} \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} g_{ij}(\tilde{x}_i) x_i^j + \sum_{1 \le i_1 < i_2 \le n} \sum_{j_1=0}^{\alpha_{i_1}-1} \sum_{j_2=0}^{\alpha_{i_2}-1} h_{i_1 i_2 j_1 j_2}(\tilde{x}_{i_1,i_2}) x_{i_1}^{j_1} x_{i_2}^{j_2}, \quad (6)$$

for some $h_{i_1 i_2 j_1 j_2} \in D'(\Omega)$, since multiplication by x_i preserves the $\approx_{\mathcal{D}'}$-equality. We now proceed inductively and show that

$$f(x) \approx_{\mathcal{D}'} \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} g_{ij}(\tilde{x}_i) x_i^j + \sum_{\substack{1 \le i_1 < i_2 \\ < i_3 \le n}} \sum_{j_1, j_2, j_3} h_{i_1 i_2 i_3 j_1 j_2 j_3}(\tilde{x}_{i_1,i_2,i_3}) x_{i_1}^{j_1} x_{i_2}^{j_2} x_{i_3}^{j_3}, \quad (7)$$

for some $g_{ij} \in {}^*\mathcal{C}^\infty(\Omega)$, S-continuous and finite-valued on ns(${}^*\Omega$) and some $h_{i_1 i_2 i_3 j_1 j_2 j_3} \in D'(\Omega)$.
The proof is similar. Let $F := f - \sum_{i=1}^n \sum_{j=0}^{\alpha_i - 1} g_{ij}(\tilde{x}_i) x_i^j$. Let $\alpha =: (\alpha_1, \alpha_2, \tilde{\alpha})$, $\tilde{\alpha} \in \mathbb{N}^{n-2}$. Let $\tilde{\Omega} := (a_3, b_3) \times \cdots \times (a_n, b_n)$. Then

$$\partial^{(0,0,\tilde{\alpha})} F(x) \approx_{\mathcal{D}'} \sum_{j_1=0}^{\alpha_1 - 1} \sum_{j_2=0}^{\alpha_2 - 1} \partial^{\tilde{\alpha}} h_{1,2,j_1,j_2}(\tilde{x}_{1,2}) x_1^{j_1} x_2^{j_2}.$$

Fixing now $c \in \mathrm{ns}^*(a_1, b_1)$ and $d \in \mathrm{ns}^*(a_2, b_2)$, we choose ψ_m as before, $\phi(\tilde{x}_{1,2}) \in$ Fin(${}^*\tilde{\Omega}$), multiply the previous expression by $\psi_m(c - x_1)\psi_m(d - x_2)\phi(\tilde{x}_{1,2})$ and integrate over ${}^*\Omega$ to obtain similarly that

$$\partial^{\tilde{\alpha}} F(c, d, \tilde{x}_{1,2}) \approx_{\mathcal{D}'} \sum_{j_1=0}^{\alpha_1 - 1} \sum_{j_2=0}^{\alpha_2 - 1} \partial^{\tilde{\alpha}} h_{1,2,j_1,j_2}(\tilde{x}_{1,2}) c^{j_1} d^{j_2}.$$

Now we substitute c by α_1 different values $c_1, \ldots, c_{\alpha_1} \in \mathrm{ns}^*(a_1, b_1)$ and d by α_2 different values $d_1, \ldots, d_{\alpha_2} \in \mathrm{ns}^*(a_2, b_2)$, with $c_i \not\approx c_j$ if $i \ne j$ and $d_i \not\approx d_j$ if $i \ne j$. The resulting linear system has $\alpha_1 \alpha_2$ equations and $\alpha_1 \alpha_2$ unknown functions $\partial^{\tilde{\alpha}} h_{1,2,j_1,j_2}$. The matrix of the system is (if the equations and unknowns are written down in a suitable order) the Kronecker-product (sometimes also called direct product, see e.g. [2]) of the Vandermonde-matrices $(c_i^{j-1})_{i,j=1,\ldots,\alpha_1}$ and $(d_i^{j-1})_{i,j=1,\ldots,\alpha_2}$, with determinant

$$\prod_{i<j} (c_j - c_i)^{\alpha_2} \prod_{i<j} (d_j - d_i)^{\alpha_1} \not\approx 0.$$

Another application of the previous lemma yields that

$$h_{1,2,j_1,j_2}(\tilde{x}_{1,2}) \approx_{\mathcal{D}'} g_{1,2,j_1,j_2}(\tilde{x}_{1,2}) + \sum_{i=3}^{n} \sum_{k=0}^{\alpha_i - 1} \tilde{h}_{ik}(\tilde{x}_{1,2,i}) x_i^k ,$$

for some $g_{1,2,j_1,j_2} \in {}^*\mathcal{C}^\infty(\Omega)$, S-continuous and finite-valued on $\mathrm{ns}({}^*\Omega)$ and $\tilde{h}_{ik} \in D'(\Omega)$. Substituting these expressions (for all h_{i_1,i_2,j_1,j_2}) in formula (6) and absorbing the terms $g_{i_1 i_2 j_1 j_2}(\tilde{x}_{i_1 i_2}) x_{i_1}^{j_1} x_{i_2}^{j_2}$ in the $g_{ij}(\tilde{x}_i) x_i^j$, we find formula (7). Repeatedly applying this procedure, we conclude that

$$f(x) \approx_{\mathcal{D}'} \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} g_{ij}(\tilde{x}_i) x_i^j + \sum_{j_1, j_2, \dots, j_n} c_{j_1,\dots,j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} ,$$

for some $g_{ij} \in {}^*\mathcal{C}^\infty(\Omega)$, S-continuous and finite-valued on $\mathrm{ns}({}^*\Omega)$ and constant $c_{j_1,\dots,j_n} \in D'(\Omega)$. As a constant function belonging to $D'(\Omega)$ is necessarily $\in \mathrm{Fin}({}^*\mathbb{C})$ (see theorem 11), we can absorb the terms $c_{j_1,\dots,j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ in the $g_{ij}(\tilde{x}_i) x_i^j$ and finally obtain the required formula. □

Finally, before proving our second main result, we need a lemma of Robinson's [4, Th. 5.3.14]. Robinson works with real-valued distributions on $\mathbb{R}$. We show that the result can be generalised to our situation.

Lemma 11. *Let $T \in \mathcal{D}'(\Omega)$. If there exists a representative f of T which is S-continuous at $a \in \Omega$, then $f(a) \in \mathrm{Fin}({}^*\mathbb{C})$. Moreover, the value $\mathrm{st} f(a)$ does not depend on the chosen S-continuous representative.*

Proof. Let $\varepsilon \in \mathbb{R}^+$. By S-continuity, there exists $r \in \mathbb{R}^+$ such that $|f(x) - f(a)| \le \varepsilon$, $\forall x \in {}^*B(a,r) \subseteq {}^*\Omega$. Now let $\phi \in \mathcal{D}(B(a,r))$, real-valued, $\phi(x) \ge 0$, $\forall x \in \Omega$ and $\int_\Omega \phi = 1$. Then

$$\begin{aligned} &\left| \int_{{}^*\Omega} f(x) {}^*\phi(x)\, dx - f(a) \right| \\ &= \left| \int_{{}^*B(a,r)} (f(x) - f(a)) {}^*\phi(x)\, dx \right| \le \int_{{}^*B(a,r)} \varepsilon \left| {}^*\phi(x) \right| \, dx = \varepsilon. \end{aligned}$$

As f represents T, also $|T(\phi) - f(a)| \le 2\varepsilon$. In particular, $f(a) \in \mathrm{Fin}({}^*\mathbb{C})$. For any representative g of T, S-continuous at a, we have the same inequality (possibly only for some smaller $r \in \mathbb{R}^+$), so $|f(a) - g(a)| \le 4\varepsilon$. As $\varepsilon \in \mathbb{R}^+$ arbitrarily, $\mathrm{st} f(a) = \mathrm{st} g(a)$. □

Theorem 12. *Let $f \in {}^*\mathcal{C}^\infty(\Omega)$. Then $f \approx_{\mathcal{D}'(\Omega)} 0$ iff for each $K \subset\subset \Omega$, there exists $\alpha \in \mathbb{N}^n$ and $g \in {}^*\mathcal{D}(\Omega)$ such that $g(x) \approx 0$, $\forall x \in {}^*\Omega$ and $f = \partial^\alpha g$ on ${}^*K$.*

Proof. 1. ⇒: We first consider the case where $K \subset\subset \Omega$ is an interval.
Take an interval $K' \subset\subset \Omega$ with $K \subset\subset {}^\circ(K')$. By theorem 7, there exists $h \in D'(\Omega)$ which is finite-valued and S-continuous on ${}^*K'$ and such that $\partial^\alpha h = f$ on ${}^*K'$. By lemma 10 applied on the open interval $\tilde{\Omega} := {}^\circ(K')$, we find in particular that h is $\mathcal{D}'(\tilde{\Omega})$-infinitely close to some $\tilde{h} \in {}^*\mathcal{C}^\infty(\tilde{\Omega})$, which is S-continuous on $\mathrm{ns}({}^*\tilde{\Omega})$. As $\tilde{h}(x) = \sum_{i=1}^n \sum_{j=0}^{\alpha_i - 1} g_{ij}(\tilde{x}_i) x_i^j$, we see that $\partial^\alpha \tilde{h} = 0$ on ${}^*\tilde{\Omega}$. Now $h - \tilde{h} \approx_{\mathcal{D}'(\tilde{\Omega})} 0$ and is S-continuous on $\mathrm{ns}({}^*\tilde{\Omega})$, so by lemma 11,

$h(x) - \tilde{h}(x) \approx 0$, $\forall x \in \mathrm{ns}({}^*\tilde{\Omega})$. Further, $\partial^\alpha(h - \tilde{h}) = \partial^\alpha h = f$ on ${}^*K$. If $\phi_0 \in \mathcal{D}(\tilde{\Omega})$ with $\phi_0 = 1$ on a neigbourhood of ${}^*K$, $g := (h - \tilde{h})^*\phi_0$ has the required properties.
2. The general case, as well as the $\Leftarrow$-part follow in a way similar to theorem 7.
$\square$

5 Hoskins and Sousa Pinto's theory of distributions

In [1], R. F. Hoskins and J. Sousa Pinto introduce another nonstandard theory of distributions. In this setting, nonstandard representatives of a distribution are *by definition* locally finite-order derivatives of finite-valued and S-continuous functions. By theorem 7, it now follows that representatives of distributions in the sense of Hoskins and Sousa Pinto are exactly representatives of distributions in the sense of Stroyan and Luxemburg.

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